1. The Laplace transform can be used to determine the transfer function of an LTI system and the transfer function can be used to find the response of an LTI system to an arbitrary excitation.

2. The Laplace transform exists for signals whose magnitudes do not grow any faster than an exponential in either positive or negative time.

3. The region of convergence of the Laplace transform of a signal depends on whether the signal is right- or left-sided.

4. Systems described by ordinary, linear, constant-coefficient differential equations have transfer functions in the form of a ratio of polynomials in $s$.

5. Pole-zero diagrams of a system’s transfer function encapsulate most of its properties and can be used to determine its frequency response to within a gain constant.

6. MATLAB has an object defined to represent a system transfer function and many functions to operate on objects of this type.

7. With a table of Laplace transform pairs and properties the forward and inverse transforms of almost any signal of engineering significance can be found.

8. The unilateral Laplace transform is commonly used in practical problem-solving because it does not require any involved consideration of the region of convergence and is, therefore, simpler than the bilateral form.

**EXERCISES WITH ANSWERS**

(On each exercise, the answers listed are in random order.)

**Laplace Transform Definition**

1. Starting with the definition of the Laplace transform

$$\mathcal{L}(g(t)) = G(s) = \int_{-\infty}^{\infty} g(t)e^{-st} \, dt$$

and without using the tables, find the Laplace transforms of these signals.

(a) $x(t) = e^t u(t)$
(b) $x(t) = e^{2t} \cos(200\pi t) u(-t)$
(c) $x(t) = \text{ramp}(t)$
(d) $x(t) = te^t u(t)$
Exercises with Answers

Existence of the Laplace Transform

2. Graph the pole-zero plot and region of convergence (if it exists) for these signals.
   (a) \( x(t) = e^{-st} u(t) \)
   (b) \( x(t) = e^{3t} \cos(20\pi t) u(-t) \)
   (c) \( x(t) = e^{2t} u(-t) - e^{-5t} u(t) \)

   Answers:

   \[
   \begin{align*}
   \text{ROC:} & \quad s > 0 \\
   \end{align*}
   \]

   \[
   \begin{align*}
   \text{ROC:} & \quad s > 2 \\
   \end{align*}
   \]

   \[
   \begin{align*}
   \text{ROC:} & \quad s < 2 \\
   \end{align*}
   \]

Direct Form II System Realization

3. Draw Direct Form II system diagrams of the systems with these transfer functions.
   (a) \( H(s) = \frac{1}{s+1} \)
   (b) \( H(s) = \frac{4}{s+10} \)

   Answers:

   \[
   \begin{align*}
   X(s) & \rightarrow \left( \begin{array}{c}
   10 \\
   \end{array} \right) \rightarrow \left( \begin{array}{c}
   12 \\
   \end{array} \right) \rightarrow S^{-1} \rightarrow Y(s) \\
   \end{align*}
   \]

   \[
   \begin{align*}
   X(s) & \rightarrow \left( \begin{array}{c}
   4 \\
   \end{array} \right) \rightarrow \left( \begin{array}{c}
   3 \end{array} \right) \rightarrow Y(s) \\
   \end{align*}
   \]

Forward and Inverse Laplace Transforms

4. Using the time-shifting property, find the Laplace transform of these signals.
   (a) \( x(t) = u(t) - u(t - 1) \)
   (b) \( x(t) = 3e^{-3(t+2)} u(t+2) \)
   (c) \( x(t) = 3e^{-3t} u(t - 2) \)
   (d) \( x(t) = 5 \sin(\pi(t-1)) u(t-1) \)

   Answers:

   \[
   \begin{align*}
   & \frac{3e^{-2s-6}}{s+3}, \quad \frac{1-e^{-s}}{s}, \quad \frac{5\pi e^{-s}}{s^2 + \pi^2}, \quad \frac{3e^{2s}}{s+3} \\
   \end{align*}
   \]

5. Using the complex-frequency-shifting property, find and graph the inverse Laplace transform of
   \[
   X(s) = \frac{1}{s+4} + \frac{1}{s-j4}, \quad \sigma > -3.
   \]
6. Using the time-scaling property, find the Laplace transforms of these signals.
(a) \( x(t) = \delta(4t) \)
(b) \( x(t) = u(4t) \)
Answers: \( 1/s, \sigma > 0 \), \( 1/4 \), All \( s \)

7. Using the time-differentiation property, find the Laplace transforms of these signals.
(a) \( x(t) = \frac{d}{dt}(u(t)) \)
(b) \( x(t) = \frac{d}{dt}(e^{-10t} u(t)) \)
(c) \( x(t) = \frac{d}{dt}(4 \sin(10\pi t) u(t)) \)
(d) \( x(t) = \frac{d}{dt}(10 \cos(15\pi t) u(t)) \)
Answers: \( \frac{40\pi s}{s^2 + (10\pi)^2}, \text{Re}(s) > 0 \), \( \frac{10s^2}{s^2 + (15\pi)^2}, \text{Re}(s) > 0 \), 1, All \( s \), \( \frac{s}{s+10}, \text{Re}(s) > -10 \)

8. Using the convolution in time property, find the Laplace transforms of these signals and graph the signals versus time.
(a) \( x(t) = e^{-t} u(t) * u(t) \)
(b) \( x(t) = e^{-t} \sin(20\pi t) u(t) * u(-t) \)
(c) \( x(t) = 8 \cos(\pi t/2) u(t) * [u(t) - u(t - 1)] \)
(d) \( x(t) = 8 \cos(2\pi t) u(t) * [u(t) - u(t - 1)] \)
Answers:

9. Using the integral property (possible) of Laplace transforms.
(a) \( X(s) = \frac{2}{s} \)
(b) \( X(s) = \frac{1}{s} \)
(c) \( X(s) = \frac{1}{s} \)
(d) \( X(s) = \frac{1}{s} \)
(e) \( X(s) = \frac{1}{s} \)
Answers: 10, \( \sigma > 0 \), Does not exist.

10. Find the inverse Laplace transform.
(a) \( x(t) = \frac{s}{s^2 + 1} \)
(b) \( x(t) = \frac{s}{s^2 + 9} \)
(c) \( x(t) = \frac{s}{s^2 + 9} \)
(d) \( x(t) = \frac{s}{s^2 + 9} \)
(e) \( x(t) = \frac{s}{s^2 + 9} \)
Answers: \( 2e^{-t} \sin(3t) \), \( 2e^{-t} \sin(3t) \), \( 2e^{-t} \sin(3t) \), \( 2e^{-t} \sin(3t) \), \( 2e^{-t} \sin(3t) \)

11. Let the function be the sum of \( \delta(t) - 3e^{-3t} u(t) \) and \( e^{-2t}(1 - 2t) u(t) \).
(a) What is the Laplace transform of the function?
(b) What is the Laplace transform of the function?
Answers: An inverse Laplace transform.

Unilateral Laplace Transform

12. Starting with the unilateral Laplace transform method, find the Laplace transforms of the following signals and graph the signals versus time.

\[ x(t) = \begin{cases} 0.025 & \text{for } 0 \leq t \leq 5 \\ 0 & \text{otherwise} \end{cases} \]

\[ x(t) = \begin{cases} 2 & \text{for } 0 \leq t \leq 1 \\ 0 & \text{otherwise} \end{cases} \]

\[ x(t) = \begin{cases} 10 & \text{for } 0 \leq t \leq 8 \\ 0 & \text{otherwise} \end{cases} \]

and without using a...
9. Using the initial and final value theorems, find the initial and final values (if possible) of the signals whose Laplace transforms are these functions.

(a) \( X(s) = \frac{10}{s + 8}, \quad \sigma > -8 \)

(b) \( X(s) = \frac{s + 3}{(s + 3)^2 + 4}, \quad \sigma > -3 \)

(c) \( X(s) = \frac{s}{s^2 + 4}, \quad \sigma > 0 \)

(d) \( X(s) = \frac{10s}{s^2 + 10s + 300}, \quad \sigma < -5 \)

(e) \( X(s) = \frac{8}{s(s + 20)}, \quad \sigma > 0 \)

(f) \( X(s) = \frac{8}{s^2(s + 20)}, \quad \sigma > 0 \)

Answers: 10, Does not apply, 0, 1, 0, 0, Does not apply, 2/5, 1, Does not apply, 0, Does not apply

10. Find the inverse Laplace transforms of these functions.

(a) \( X(s) = \frac{24}{s(s + 8)}, \quad \sigma > 0 \)

(b) \( X(s) = \frac{20}{s^2 + 4s + 3}, \quad \sigma < -3 \)

(c) \( X(s) = \frac{5}{s^2 + 6s + 73}, \quad \sigma > -3 \)

(d) \( X(s) = \frac{10}{s(s^2 + 6s + 73)}, \quad \sigma > 0 \)

(e) \( X(s) = \frac{4}{s^2(s^2 + 6s + 73)}, \quad \sigma > 0 \)

(f) \( X(s) = \frac{2s}{s^2 + 2s + 13}, \quad \sigma < -1 \)

(g) \( X(s) = \frac{s}{s^2 + 3}, \quad \sigma > -3 \)

(h) \( X(s) = \frac{s}{s^2 + 4s + 4}, \quad \sigma > -2 \)

(i) \( X(s) = \frac{s^2}{s^2 - 4s + 4}, \quad \sigma < 2 \)

(j) \( X(s) = \frac{10s}{s^4 + 4s^2 + 4}, \quad \sigma > -2 \)

Answers: \( 2e^{-t}[(1/\sqrt{12})\sin(\sqrt{12}t) - \cos(\sqrt{12}t)]u(-t), 10(e^{-3t} - e^{-t})u(-t), \)

\( e^{-2t}(1 - 2t)u(t), \frac{10}{73}[1 - \sqrt{73}/64e^{-3t}\cos(8t - 0.3588)]u(t), 5(e^{-2t}(t + 1) - 3e^{-3t}u(t), 3(1 - e^{-8t})u(t), (5/\sqrt{2})e^{-3t}\sin(\sqrt{2}t)u(t)] \)

11. Let the function \( x(t) \) be defined by \( x(t) = \frac{s(s + 5)}{s^2 + 16}, \quad \sigma > 0. \) \( x(t) \) can be written as the sum of three functions, two of which are causal sinusoids.

(a) What is the third function?

(b) What is the cyclic frequency of the causal sinusoids?

Answers: An impulse, 0.637 Hz.

Unilateral Laplace Transform Integral

12. Starting with the definition of the unilateral Laplace transform

\[
L(g(t)) = G(s) = \int_{0}^{\infty} g(t)e^{-st} dt
\]

and without using any tables, find the unilateral Laplace transforms of these signals.
(a) \( x(t) = e^{-t} u(t) \)
(b) \( x(t) = e^{2t} \cos(200\pi t) u(t) \)
(c) \( x(t) = u(t+4) \)
(d) \( x(t) = u(t-4) \)

Answers: \( \frac{1}{s+1}, \sigma > 1, \frac{1}{s}, \sigma > 0, \frac{s-2}{(s-2)^2 + (200\pi)^2}, \sigma > 2, \frac{e^{-4s}}{s}, \sigma > 0 \)

Solving Differential Equations

13. Using the unilateral Laplace transform, solve these differential equations for \( t \geq 0 \).

(a) \( x'(t) + 10x(t) = u(t), \quad x(0^-) = 1 \)

(b) \( x''(t) - 2x'(t) + 4x(t) = u(t), \quad x(0^-) = 0, \quad \left[ \frac{d}{dt} x(t) \right]_{t=0^-} = 4 \)

(c) \( x'(t) + 2x(t) = \sin(2\pi t) u(t), \quad x(0^-) = -4 \)

Answers: \( \frac{1}{4}(1 - e^t \cos(\sqrt{3}t) + (1/\sqrt{3})e^t \sin(\sqrt{3}t)) u(t), \) \( \frac{1 + 9e^{-10t}}{10} u(t), \)

\( x(t) = \left[ \frac{2\pi e^{-2t} - 2\pi \cos(2\pi t) + 2\sin(2\pi t) - 4e^{-2t}}{4 + (2\pi)^2} \right] u(t) \)

14. Write the differential equations describing the systems in Figure E.14 and find and graph the indicated responses.

(a) \( x(t) = u(t), y(t) \) is the response, \( y(0^-) = 0 \)

(b) \( v(0^-) = 10, v(t) \) is the response

Answers:

\[ y(t) \]
\[ 0.25 \]
\[ t \]
\[ v(t) \]
\[ 10 \]
\[ 0.001 \]
Pole-Zero Diagrams and Frequency Response

15. For each pole-zero diagram in Figure E.15 sketch the approximate frequency response magnitude.

(a) \[ \omega \] \hfill (b) \[ \omega \] \hfill (c) \[ \omega \] 

(a) \[ \sigma \] \hfill (b) \[ \sigma \] \hfill (c) \[ \sigma \] 

(d) \[ \sigma \] \hfill (e) \[ \sigma \] 

Figure E.15

Answers:

[Graphs of frequency response magnitudes for each pole-zero diagram]
Chapter 8  The Laplace Transform

EXERCISES WITHOUT ANSWERS

Laplace Transform Definition

16. Using the integral definition find the Laplace transform of these time functions.
   (a) \( g(t) = e^{-at} u(t) \)
   (b) \( g(t) = e^{-a(t-\tau)} u(t-\tau) \)
   (c) \( g(t) = \sin(\omega_0 t) u(-t) \)
   (d) \( g(t) = \text{rect}(t) \)
   (e) \( g(t) = \text{rect}(t - 1/2) \)

Existence of the Laplace Transform

17. Graph the pole-zero plot and region of convergence (if it exists) for these signals.
   (a) \( x(t) = e^{-t} u(-t) - e^{-4t} u(t) \)
   (b) \( x(t) = e^{-2t} u(-t) - e^{t} u(t) \)

Direct Form II System Realization

18. Draw Direct Form II system diagrams of the systems with these transfer functions.
   (a) \( H(s) = \frac{s^2 + 8}{s^3 + 3s^2 + 7s + 22} \)
   (b) \( H(s) = \frac{s + 20}{(s + 4)(s + 8)(s + 14)} \)

Forward and Inverse Laplace Transforms

19. Using a table of Laplace transforms and the properties find the Laplace transforms of the following functions.
   (a) \( g(t) = 5 \sin(2\pi(t - 1)) u(t - 1) \)
   (b) \( g(t) = 5 \sin(2\pi t) u(t - 1) \)
   (c) \( g(t) = 2 \cos(10\pi t) \cos(100\pi t) u(t) \)
   (d) \( g(t) = \frac{d}{dt} (u(t - 2)) \)
   (e) \( g(t) = \int_0^t u(\tau) d\tau \)
   (f) \( g(t) = \frac{d}{dt} (5e^{-(t-\tau)/2} u(t-\tau)), \tau \geq 0 \)
   (g) \( g(t) = 2e^{-5t} \cos(10\pi t) u(t) \)
   (h) \( x(t) = 5 \sin(\pi t - \pi/8) u(-t) \)

20. Given

\[ g(t) \xrightarrow{L} \frac{s + 1}{s(s + 4)}, \sigma > 0 \]

find the Laplace transforms of
   (a) \( g(2t) \)
   (b) \( \frac{d}{dt} (g(t)) \)
   (c) \( g(t - 4) \)
   (d) \( g(t) * g(t) \)

21. Find the functions with the
   (a) \( G(s) \)
   (b) \( G(s) \)
   (c) \( G(s) \)

22. Given

   find the initial
   (a) \( G(s) \)
   (b) \( G(s) \)

23. Find the

24. A system

   partial fr

   the num

Solution of DE

25. Write the

   and grap

   (a) \( x(t) \)
   (b) \( i_s(t) \)
21. Find the time-domain functions that are the inverse Laplace transforms of these functions. Then, using the initial and final value theorems, verify that they agree with the time-domain functions.

(a) \( G(s) = \frac{4s}{(s+3)(s+8)} , \sigma > -3 \)

(b) \( G(s) = \frac{4}{(s+3)(s+8)} , \sigma > -3 \)

(c) \( G(s) = \frac{s}{s^2 + 2s + 2} , \sigma > -1 \)

(d) \( G(s) = \frac{e^{-2s}}{s^2 + 2s + 2} , \sigma > -1 \)

22. Given \( e^{4t} u(-t) \xrightarrow{L} G(s) \)

find the inverse Laplace transforms of

(a) \( G(s/3) , \sigma < 4 \)

(b) \( G(s - 2) + G(s + 2) , \sigma < 4 \)

(c) \( G(s)/s , \sigma < 4 \)

23. Find the numerical values of the constants \( K_0 , K_1 , K_2 , p_1 \) and \( p_2 \).

\[
\frac{s^2 + 3}{3s^2 + s + 9} = K_0 + \frac{K_1}{s - p_1} + \frac{K_2}{s - p_2}
\]

24. A system has a transfer function \( H(s) = \frac{s(s-1)}{(s+2)(s+a)} \), which can be expanded in partial fractions in the form \( H(s) = A + \frac{B}{s+2} + \frac{C}{s+a} \). If \( a \neq 2 \) and \( B = 3/2 \), find the numerical values of \( a, A \) and \( C \).

Solution of Differential Equations

25. Write the differential equations describing the systems in Figure E.25 and find and graph the indicated responses.

(a) \( x(t) = u(t) \), \( y(t) \) is the response, \( y(0^-) = -5 \), \( \left[ \frac{d}{dt} (y(t)) \right]_{t=0^-} = 10 \)

(b) \( i_x(t) = u(t) \), \( v(t) \) is the response, No initial energy storage
Chapter 8  The Laplace Transform

(c) \( i_v(t) = \cos(2000\pi t)u(t) \), \( v(t) \) is the response, No initial energy storage

Figure E.25

Pole-Zero Diagrams and Frequency Response

26. Draw pole-zero diagrams of these transfer functions.

(a) \( H(s) = \frac{(s + 3)(s - 1)}{s(s + 2)(s + 6)} \)  
(b) \( H(s) = \frac{s}{s^2 + s + 1} \)

(c) \( H(s) = \frac{s(s + 10)}{s^2 + 11s + 10} \)  
(d) \( H(s) = \frac{1}{(s + 1)(s^2 + 1.618s + 1)(s^2 + 0.618s + 1)} \)

Answers:

27. In Figure E.27 are some pole-zero plots of transfer functions of systems of the general form,

\[ H(s) = \frac{(s - z_1) \cdots (s - z_N)}{(s - p_1) \cdots (s - p_D)} \]

in which \( A = 1 \), the \( z \)'s are the zeros and the \( p \)'s are the poles. Answer the following questions.

(a) Which one(s) have a magnitude frequency response that is nonzero at \( \omega = 0 \)?

(b) Which one(s) have a magnitude frequency response that is nonzero as \( \omega \to \infty \)?

(c) There are two that have a bandpass frequency response (zero at zero and zero at infinity). Which one is more underdamped?

(d) Which one has a magnitude frequency response whose shape is closest to being a bandstop filter?

(e) Which one(s) have a magnitude frequency response that approaches \( K/\omega^6 \) at very high frequencies (\( K \) is a constant)?

(f) Which one has a magnitude frequency response that is constant?

(g) Which one(s) have a magnitude frequency response whose shape is closest to lowpass filter?
(h) Which one(s) have a phase frequency response that is discontinuous at \( \omega = 0 \)?

![Figure E.27](image-url)

For each of the pole-zero plots in Figure E.28 determine whether the frequency response is that of a practical lowpass, bandpass, highpass or bandstop filter.

![Figure E.28](image-url)